# Regime Dependent Effects of Monetary and Fiscal Policy on the Distribution of Inflation Expectations

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**Abstract**

Regime switching models are a popular tool to study asymmetric dynamics in economic variables, for example, the business cycle, government spending multipliers, or the interaction between monetary and fiscal policy. This paper introduces a regime-switching model for functional time series. We apply our model to the expected inflation distribution (EID), representing consumers' heterogeneous beliefs about future inflation. For this time series of density functions, we identify volatile and stable regimes based on the magnitude of changes across all aspects of EIDs. In the volatile regime, a contractionary shock of twenty-five b.p. increases the mean one-year inflation expectation by eight b.p. while increasing disagreement by four b.p. and reducing inflation expectations anchoring; in the stable regime, the mean increases by three b.p. without effects on dispersion or expected inflation anchoring. An increase in government spending shows a mild uptick in the average inflation expectations only in the stable regime.

JEL CLASSIFICATION: C32; C54; E42; E52; E62; E63

KEY WORDS: Expected Inflation, Expected Inflation Disagreement, Functional Time Series, Regime-Switching Model, Monetary Policy, Fiscal Policy

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# 1 Introduction

The recurrent need in economics to measure asymmetric relationships and the growing research using functional data has motivated us to study a regime-switching model for functional time series.

Functional models have become popular, mainly due to their efficient representation of cross-sectional and term structure data. The yield curve, heterogeneous inflation expectations, expected inflation term structure, income and consumption distributions, global temperatures, and demographics are examples of variables studied using functional time series.

We use the expected inflation distribution (EID) as an example application for our model. In analyzing the dynamic effects of economic shocks on this distribution, Chang, Gómez-Rodríguez, and Hong (2022) observed significant changes in the magnitude of the period-by-period variations of its density. To illustrate this behavior, we took several statistics describing survey data on expected inflation[1], calculated the yearly variance of these statistics, and plotted them in figure 1. Periods with changes of increased magnitude are evident for all statistics, suggesting that it is the entire distribution having a regime-switching behavior.

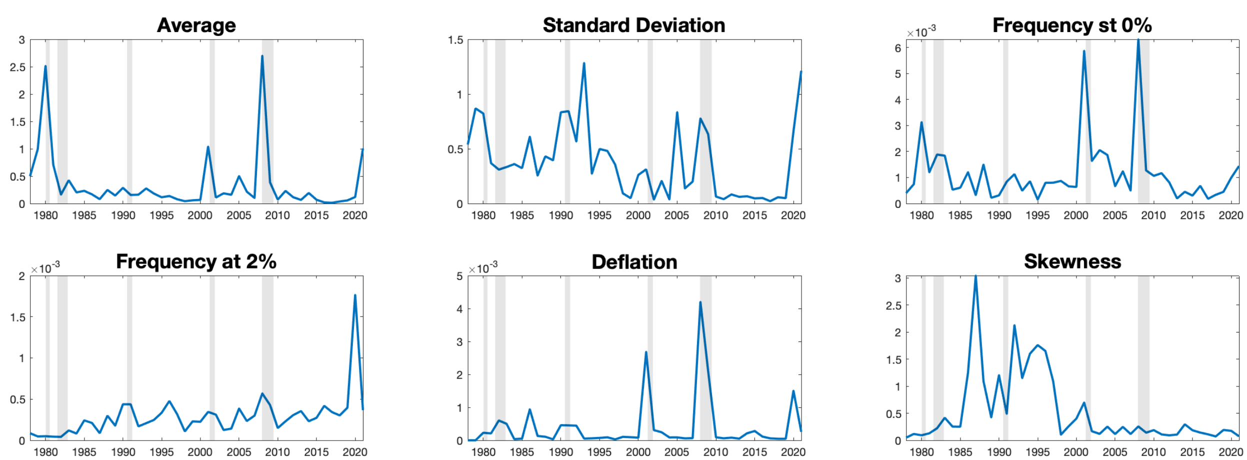


***Figure 1.*** *Variance of sample statistics of the EID distribution.*

*Note:* Estimation of the calendar year variance of the sample average, standard deviation, relative frequency of 0% and 2% inflation expectations, portion of deflation expectations and sample skewness of the responses of the University of Michigan's Survey of Consumers. Sample from January 1978 to December 2021. For reference, shaded areas represent the NBER Recessions.

[1]Survey of Consumers by the University of Michigan

With our model, we analyze whether the effects of policy on inflation expectations are regime dependent. We identify two regimes: *stable* and *volatile*, based on the size of the EID variations. Note that our regimes are based on the time-variance of the entire distribution (all its features) and not the cross-sectional variance (or standard deviation) which is only one aspect of EID.

In the volatile regime, a contractionary shock of twenty-five b.p. increases the mean one-year inflation expectation by eight b.p. while increasing disagreement by four b.p. and reducing inflation expectations anchoring; in the stable regime, the mean increases by three b.p. without effects on dispersion or expected inflation anchoring. An increase in government spending shows a mild uptick in the average inflation expectations only in the stable regime.

Our *functional endogenous regime-switching model* is an extension of the model proposed by Chang, Choi, and Park (2017) (CCP). The key steps to adapt their model to *functional time series* are: i) use functional principal components to model the dynamics of the functional process with a vector autoregression model[2], and ii) adapt CCP's scalar AR model to a multivariate set-up and apply it to the VAR model obtained in the first step. The latter step is the main contribution of this paper.

Functional principal components (FPC) represent densities as a linear combination of just a few time-invariant basis functions. We stack the loadings of the linear combinations in a vector and use a VAR to represent their dynamics. Based on their shape we get to interpret the first FPC as *disagreement*, *shifting* and *ambiguity* components.

Based on the VAR of loadings, we consider a regime-switching model where the covariance matrix is regime-dependent. To identify the regimes, we focus on the largest eigenvalue of the covariance matrix. For the regime dynamics, we assume there is a latent factor that, combined with a threshold parameter, determines the model's regime. We allow the innovations to the functional time series to influence the latent factor with an endogenous feedback element. The latter makes time-varying regime transition probabilities possible, a feature that typical Markov regime-switching models do not have. Besides inferred probabilities and regime-dependent IRFs, our model also extracts the expected value of the latent factor.

[2]The statistical properties of the latter step is rigorously studied by Chang, Park, and Pyun (2021). We refer interested readers to this paper.

Our application uses densities estimated using data from the Survey of Consumers by the University of Michigan. The focus is on expected price changes over the following twelve months, specifically on the responses to the question: "*About what percent do you expect prices to go (up/down) on the average, during the next 12 months?*". This question is one of 50 core questions in the survey. The estimation is non-parametric and makes no strong assumptions on the structure of expected inflation's distribution.

We use the correlation of external policy shocks (from the literature) and functional, structural shocks obtained from the RS-VAR model to determine the distribution of inflation expectations to policy shocks.

**Literature Review.** This paper contributes to several strands of the literature, for instance, to functional methods in macroeconomics. Inoue and Rossi (2019) use functional data to identify monetary policy shocks. They do that based on the response of the entire yield curve to policy announcements. Chang, Gómez-Rodríguez, and Matthes (2020) study the effects of the U.S. government decisions on its borrowing costs. Based on long-run restrictions, our analysis focuses on the yield curve's response to fiscal policy measures.

Chang, Kim, and Park (2018) (CKP) and Chang, Chen, and Schorfheide (2018) independently consider a functional autoregression that stacks macroeconomic aggregates and a cross-sectional density. Both papers apply their econometric model to study the effects of macroeconomics shocks on income distribution. Chang, Miller, and Park (2019) applies the functional SVAR methodology introduced by CKP on the time series of key aggregate climate variables and global temperature distributions to study the effects of various popular natural and anthropogenic shocks to the climate system.

Additionally, this paper contributes to the literature on expected inflation disagreement. Mankiw, Reis, and Wolfers (2004) argues that disagreement contributes to the analysis of monetary policy and the business cycle. Other papers analyze the real effects of expected inflation disagreement. For example, Ehling et al. (2015) show that inflation disagreement drives a wedge between real and nominal yields. Falck, Hoffmann, and Hürtgen (2019) show that monetary policy effectiveness is regime-dependent, based on the cross-sectional standard deviation of inflation expectations.

Using the framework we used here a starting point, Chang, Gómez-Rodríguez, and Hong (2022) study the effects of multiple economic shocks on the distribution of inflation expectations. We extend the analysis to incorporate regime contingent effects.

**Outline.** Section two goes through the mathematical and technical details of functional time series. We describe the conditions and restrictions we impose on functional time series to formulate the regime-switching model. In section three we describe the functional endogenous regime-switching model and the procedure to obtain functional IRFs from external shocks. Our application case begins with section four, which characterizes the process that estimates the densities from the survey data. Section five analyzes the estimation results and the regime-dependent effects of fiscal and monetary policy. Section six concludes.

## 2 Modelling Functional Time Series

This section will lay the groundwork for the formal analysis of functional time series. It describes the type of mathematical space we use for the functions in the time series: *separable Hilbert spaces*. Then, this section presents functional principal components to determine the finite-dimensional basis that summarizes the functional time series in a small set of components explaining most of the process' total variance.

### 2.1 Separable Hilbert Space

We assume that the functional time series takes values on $H = L^2(\mathbb{R})$, the space of square-integrable functions. That is, in the space $H$ we will find all functions $u : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\int_{\mathbb{R}} u^2(\tau) d\tau < \infty$$

The space $H$ is a *Hilbert space*, which means that an *inner product* can be defined for functions in it. We define the following inner product for any two functions $u$ and $v$ in $H$

$$\langle u, v \rangle = \int_{\mathbb{R}} u(\tau) v(\tau) d\tau$$

It is a well-known fact that for $L^2$-functions, such as the ones in $H$ this inner product is well-defined. The norm induced by this inner product is defined by $\|u\| = \sqrt{\langle u, u\rangle}$.

With an inner product we gain geometric intuition to analyze functions. We can talk of length of a function and orthogonality between two functions. This will be helpful to construct functional principal components.

The Hilbert space $H$ is also *separable*. This means that it has a countable basis. For an (orthonormal) basis of H $(v_i)$ any element $f$ of $H$ may be expressed as a linear combination of basis functions

$$f = \sum_{i=1}^{\infty} \langle v_i, f\rangle v_i$$

in $L^2$-sense. With a truncation number $m$ we may approximate the function $f$ with

$$f \approx \sum_{i=1}^{m} \langle v_i, f\rangle v_i \tag{1}$$

The tensor product $u \otimes w$ with any given $u$ and $w$ in $H$ is a linear operator on $H$ defined as $(u \otimes w)v = \langle v, w\rangle u$ for all $v$ in $H$. If $H$ is finite dimensional and represented by $\mathbb{R}^n$, $u \otimes w = uw'$, i.e., $u \otimes w$ reduces to the outer product, contrastingly with the inner product $\langle u, w\rangle = u'w$, where $u'$ and $w'$ are the transposes of $u$ and $w$. If $u$ and $w$ are random functions taking values in $H$, then their covariance operator $\mathbb{E}(u \otimes w)$ is generally defined as a linear operator satisfying

$$\left\langle v_i, [\mathbb{E}(u \otimes w)]v_j \right\rangle = \mathbb{E}\langle v_i, u\rangle\langle v_j, w\rangle$$

for all $v_i$ and $v_j$ in $H$.

A critical step to move forward is to choose a basis for the Hilbert space $H$. With this goal in mind, we use functional principal components to define the basis $\{v_1^*, v_2^*, \ldots\}$ in $H$.

### 2.2 Functional Principal Component Basis

Typical examples of functional bases are given by sets of deterministic functions such as polynomials and trigonometric functions. They are independent of the data. In contrast, the functional basis used in this paper is obtained by functional principal components analysis, and determined directly by the actual data. The use of any deterministic functional basis, at least theoretically, is allowed and legitimate. However, from the practical point of view, it is very important to choose a functional basis whose leading components effectively summarize the time series variation.

Let $f_t \in H$ represent the functional time series we would like to analyze. Denote by $\Gamma$ the sample variance of $f_t$, i.e.,

$$\Gamma = T^{-1} \sum_{t=1}^{T} (f_t \otimes f_t)$$

and by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ its ordered eigenvalues, and by $v_1^*$,$v_2^*$, ... the corresponding orthonormal eigen-functions. These are the components that build the principal component basis. The eigenvalues are interpreted as the portion of the variance explained by the corresponding eigenfunctions. Note that $\Gamma$ is a self-adjoint operator on $H$, and therefore, its eigenvalues are all real.

The cumulative portion of the variance explained by the $m$ leading factors, $v_1^*, v_2^*, v_3^*, \ldots, v_m^*$, is given by

$$\theta(m) = \frac{\sum_{i=1}^{m} \lambda_i}{\sum_{i=1}^{\infty} \lambda_i}. \tag{2}$$

Note that $\Gamma$ has only $T$ nonzero eigenvalues, and therefore, $\lambda_i = 0$ for all $i > T$.

**Example: EID.** To illustrate this, in figure 2 we present the so-called scree-plot, with the portion $\theta(m)$ of variance explained by the first five components for our example functional time series the EID. In the implementation part, we use $m = 3$ since it is the first value of $m$ whose $\theta(m)$ surpasses 95% of the total variance. See Paparoditis et al. (2018) for a justification of this approach to choose $m$. There is no strict rule that can be used to determine $m$. The approximation involved in representing infinite dimensional functions by finite dimensional vectors requires $m \to \infty$ as $T \to \infty$, and the existing theory relevant to the choice of $m$ only specifies at what rate $m$ diverges to infinity as $T$ increases. The main results in the paper are not sensitive to the choice of $m$, unless it is set to be too large.[3]

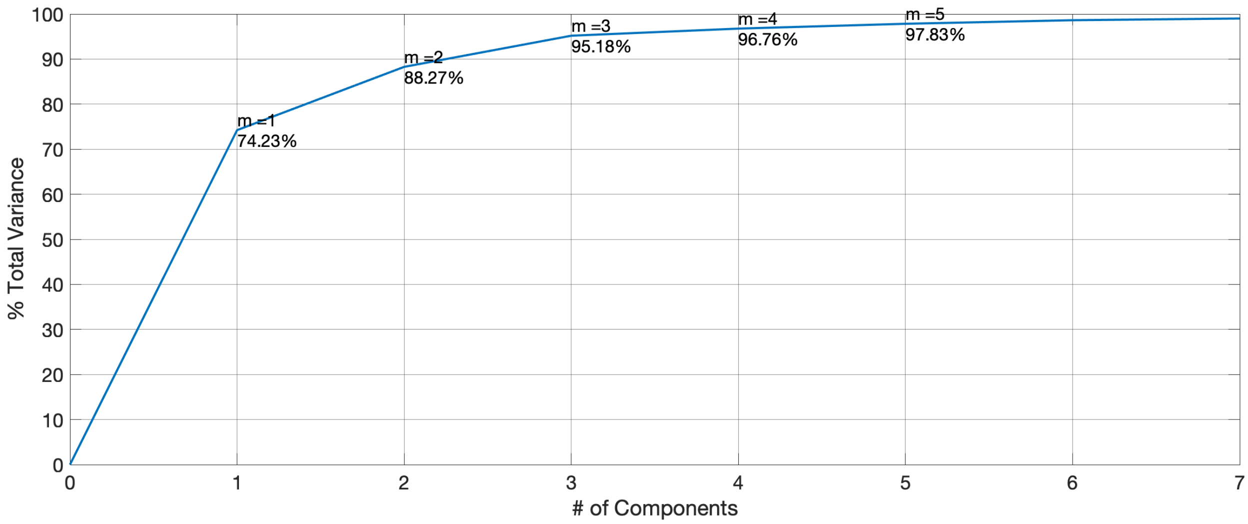


***Figure 2.*** *Scree-plot: What portion of the variance of EIDs is explained by FPC?*

*Note:* Proportion of the variance of the time series of EIDs which is explained by the functional principal components. Three principal components explain above 95% of the variability of the functional process.

To illustrate the principal components figure 3, 4 and 5 show and explain $v_1, v_2$ and $v_3$ together with its respective loadings. Together they explain over 95% of the process's total variance. Each of the figures show the shape of the functional component, the "range of motion" generated by this component and the loading in the sample.

**Interpretation of Functional Principal Components** We gave the name *disagreement* to the first component, positive values on this direction will reduce inflation expectations between -2% and 6% and increase deflation expectations below -2% and above 6%. This increases the standard deviation of the distribution. For the second component we use the name *shift*, since it lowers the density of inflation expectations below 2% and increases inflation expectations above. Finally, the ambiguity component owns its name to the fact that inflation expectations at 2% (stable inflation) and high inflation expectations increase their density.

[3]The main empirical results in the paper are obtained from a structural VAR with regime switching. It is therefore natural to expect that they become unstable if based on excessively large dimensional VARs.

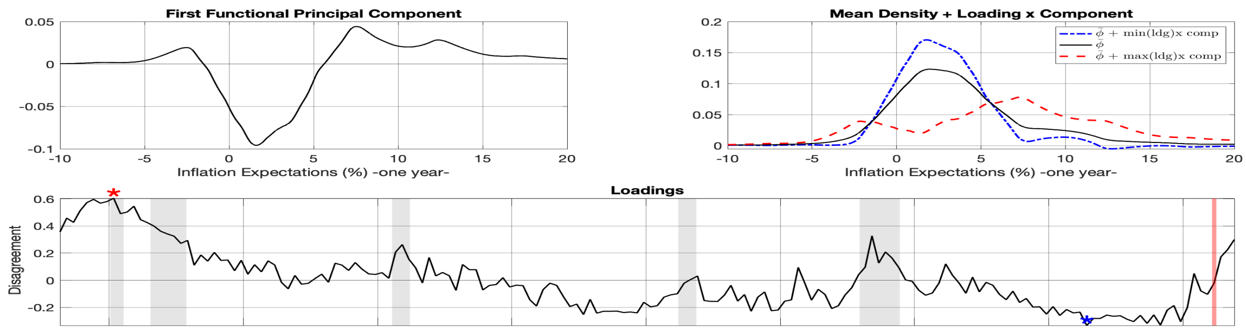


***Figure 3.*** *Functional Principal Component: Disagreement*

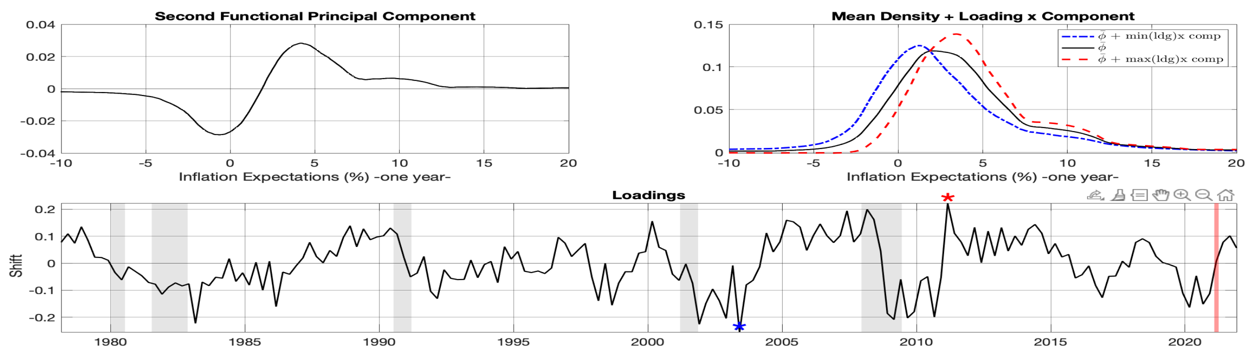


***Figure 4.*** *Functional Principal Component: Shift*

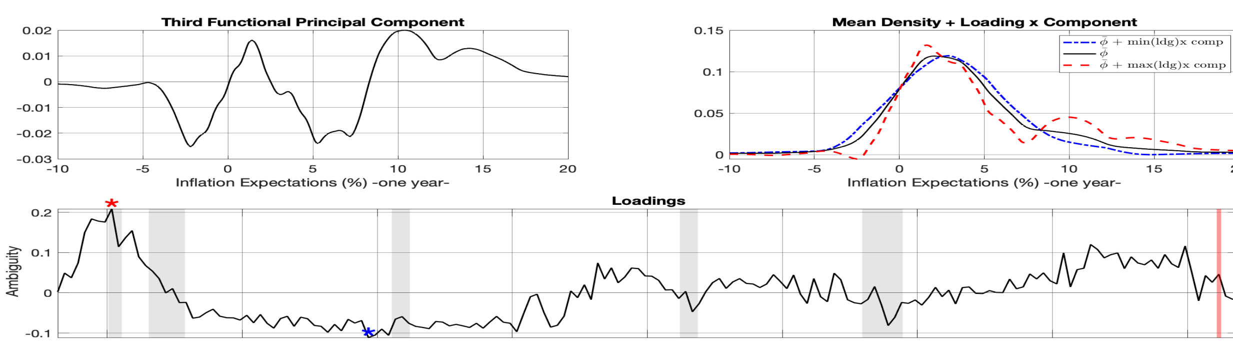


***Figure 5.*** *Functional Principal Component: Ambiguity*

### 2.3 From Functions to Vectors

Once a number of principal components is fixed, we can use a vector to represent a the density function as well as matrices to represent linear operators in $H$.

For a given truncation number $m$, define a mapping

$$\pi : f \mapsto \begin{pmatrix} \langle v_1, f \rangle \\ \vdots \\ \langle v_m, f \rangle \end{pmatrix} \tag{3}$$

and write

$$\pi(f) = \lbrace f \rbrace \tag{4}$$

for any $f$ in $H$. Accordingly, for bounded linear operators[4] let

$$\pi : A \mapsto \begin{pmatrix} \langle v_1, Av_1 \rangle & \cdots & \langle v_1, Av_m \rangle \\ \vdots & \vdots & \vdots \\ \langle v_m, Av_1 \rangle & \cdots & \langle v_n, Av_m \rangle \end{pmatrix} \tag{5}$$

be the corresponding mapping on $A$ given by

$$\pi(A) = \lbrace A \rbrace \tag{6}$$

for any operator $A$ on $H$.

It is important to clarify, that $\lbrace f \rbrace$ is a vector whose coordinates are not to be interpreted, instead, they are the weights of the principal components $v_1^*, v_2^*, \cdots, v_m^*$, to recover the approximated function of $f_t$ (see equation 1).

Our next step is to model the dynamics of functional time series $f_t$ using a regime switching vector autoregression on $\lbrace f_t \rbrace$.

[4]Linear operators mapping elements of $H$ into itself.

# 3 Modelling Functional Time Series Dynamics

This section discusses the econometric framework used to analyze the dynamics of the functional time series. First, we describe the regime-switching model that will represent the dynamics of the process. Followed by the procedure used to obtain the response of $f_t$ to the external monetary and fiscal policy shocks.

## 3.1 The Regime Switching Model

The time series of basic statistics displayed in Figure 9 made it clear that the time series of expected inflation densities has periods of high volatility and periods of low volatility. This observation motivates the introduction of a regime-switching model for functional time series.

We model the distribution of inflation expectations to switch between a *volatile* regime and a *stable* regime. It is reasonable to expect that any factor influencing the process's regime should also directly affect the variable that determines regime switching. For this reason, an endogenous regime-switching model like the one introduced in Chang, Choi, and Park (2017) is ideal in this context. A consequence of incorporating endogeneity into the model is that the transition probabilities are time-varying[5]. My contribution is to adapt the model to functional data.

The model has three parts: i) the regime dependent autoregressive model, ii) a latent variable that dictates the regime of the process, and iii) the endogenous feedback channel from the innovations to the expected inflation densities to the regime transition probabilities.

The functional regime switching model used in the paper is specified as

$$(f_t) = (A)(f_{t-1}) + B(s_t)e_t, \tag{7}$$

where $\varphi_t$ is the time series of demeaned expected inflation densities modelled as a functional autoregressive process with the autoregressive operator $A$ and the error volatility operator $B(s_t)$ given as a function of the state process $s_t$ taking values 1 and 0, which refer to the volatile and stable regimes, respectively. The presence of regime switching allows the volatility changes in

[5]Classical Markov regime switching models have time-invariant transition probabilities.

the dynamics of the distribution of inflation expectations. The operators $A$, $B(1)$ and $B(0)$ are all assumed to be compact, and the innovation $e_t$ is defined as a functional time series with identity covariance operator as required for the identification of $B(s_t)$.

The state process $s_t$ is defined as $s_t = 1\{w_t > \tau\}$, where $w_t$ is an autoregressive latent factor generated as

$$w_t = \alpha w_{t-1} + \eta_t \tag{8}$$

$$\eta_t = \langle \rho, e_{t-1} \rangle + \sqrt{1 - \|\rho\|^2}\, \delta_t, \tag{9}$$

where $\alpha \in (-1, 1]$ is the autoregressive coefficient of $w_t$, $\eta_t$ and $\delta_t$ are zero mean and unit variance random sequences that are independent of each other, and $\rho$ is the correlation function between $e_{t-1}$ and $\eta_t$. The autoregressive coefficient $\alpha$ determines the persistence of the regimes, and in particular values of $\alpha$ close to one indicate more persistent regimes. The regime switching model defined by (7), (8) and (9) may be viewed as an extension of the model in Chang, Choi, and Park (2017) to functional time series models.

The regime switching mechanism introduced here is endogenous, in the sense that the functional innovation $e_t$ can influence the regime transition probabilities to the next period. This feature will be referred to as the *endogenous feedback* more explicitly. The strength of endogenous feedback is determined by the magnitude of correlation $\rho$ between $e_{t-1}$ and $\eta_t$. Note that $\rho$ is a single function, since $e_{t-1}$ is a function while $\eta_t$ is a scalar. If $\|\rho\| = 0$, the endogenous feedback disappears, and the regime switching model defined above reduces to the classical Markov switching model, where regime switching is assumed to be entirely exogenous and driven by a Markov state variable. The correlation function $\rho$ determines how a shock to the process of expected inflation densities influences the transition probability. When $\langle \rho, e_{t-1} \rangle$ is positive(negative), the probability of switching to the second regime in the next period is higher(lower), or equivalently, the probability of switching to the first regime is lower(higher). In the presence of endogenous feedback, the transition probability clearly changes in time. This is in sharp contrast with the conventional Markov switching model.

For the estimation of the model, Gaussianity is assumed. The scalar innovations $\eta_t$ and $\delta_t$ are assumed to be standard normal. Likewise, $e_t$ is assumed to be a standard Gaussian process, i.e., for any function $v \in H$, $\langle v, e_t \rangle$ is normally distributed with mean zero and variance $\|v\|^2$.

### 3.1.1 Estimation

To estimate the model consider the projection on $H_m$ as described in the previous section. Let $\varphi_t^m = \pi_m(\varphi_t)$, $e_t^m = \pi_m(e_t)$, $A_m = \pi_m(A)$, $B_m(s_t) = \pi_m(B(s_t))$, and $\rho_m = \pi_m(\rho)$. Estimation is based on the regime switching VAR model

$$\varphi_t^m = A_m \varphi_{t-1}^m + B_m(s_t) e_t^m, \tag{7'}$$

which approximates (7). It is well expected from Paparoditis et al. (2018) that the approximation of (7) by (7′) is valid if $m$ is set to diverge to infinity as $T$ increases at an appropriate rate. However, the validity of using (7′) as an approximate model will not be discussed in any detail, since it is not a main focus of the paper.

In the following, we describe the estimation procedure from Chang, Choi, and Park (2017) applied to the model in this paper. The estimation is obtained from a maximum likelihood method. The log-likelihood function is given by

$$\ell(\varphi_1^m, \cdots, \varphi_T^m) = \log p(\varphi_1^m) + \sum_{t=1}^{T} \log(p(\varphi_t^m | \mathcal{F}_{t-1})),$$

where $\mathcal{F}_t$ represent the information given by $\varphi_1^m, \cdots, \varphi_t^m$ for $t = 1, 2, \ldots, T$. This log-likelihood function depends on a set of parameters denoted by $\theta$ for brevity in notation. The maximum likelihood estimator of $\theta$ is obtained as the maximizer of the log-likelihood function over the parameter set $\Theta$, i.e., $\underset{\theta \in \Theta}{\operatorname{argmax}}\, \ell(\varphi_1^m, \cdots, \varphi_T^m)$. The parameter $\theta$ includes $A_m$, $B_m(0)$, $B_m(1)$, $\alpha$, $\rho_m$ and $\tau$. The estimation of the model is performed using the modified Markov switching filter developed in Chang, Choi, and Park (2017), which we extend to the multivariate case here in this paper.

The following theorem specifies the joint transition of $(s_t)$ and $(y_t)$ in the case $\|\rho_m\| < 1$, where the norm $\|\cdot\|$ now refers to the Euclidean norm in $\mathbb{R}^m$.[6] This version of the theorem is specific to

[6]The notation $\|\cdot\|$ is abused to denote the Euclidean norm in $\mathbb{R}^m$ as well as the norm in $H$. This is to avoid introducing additional notation, and should not cause any confusion.

the case where the data generating process is of order one, but the theorem can easily be extended to higher order models.

**Theorem 1.** *Let $\|\rho_m\| < 1$. The $m+1$ dimensional process $(s_t, \varphi_t)$ on $\{0,1\} \times \mathbb{R}^m$ is a first order Markov process, whose transition density with respect to the product of the counting and Lebesgue measure is given by*

$$p(s_t, \varphi_t^m | s_{t-1}, \varphi_{t-1}^m) = p(\varphi_t^m | s_t, \varphi_{t-1}^m) \times p(s_t | s_{t-1}, s_{t-2}, \varphi_{t-1}^m),$$

*where*

$$p(\varphi_t^m | s_t, \varphi_{t-1}^m) = \mathbb{N}(A_m \varphi_{t-1}^m, B_m(s_t) B_m(s_t)')$$

*and*

$$p(s_t | s_{t-1}, \varphi_{t-1}^m, \varphi_{t-2}^m) = (1 - s_t)\omega_\rho + s_t(1 - \omega_\rho)$$

*with transition probability $\omega_\rho = \omega_\rho(s_{t-1}, \varphi_{t-1}^m, \varphi_{t-2}^m)$ of $(s_t)$ to the state $s_t = 0$ conditional on the previous states and the past values of observed time series. If $|\alpha| < 1$,*

$$\omega_\rho = \begin{cases} \dfrac{\int_{-\infty}^{\tau\sqrt{1-\alpha^2}} \Phi\left(\frac{\tau - \rho_m' e_{t-1}^m}{\sqrt{1-\|\rho_m\|^2}} - \frac{\alpha x}{\sqrt{1-\alpha^2}\sqrt{1-\|\rho_m\|^2}}\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\alpha^2})} & \text{if } s_{t-1} = 0 \\ \dfrac{\int_{\tau\sqrt{1-\alpha^2}}^{\infty} \Phi\left(\frac{\tau - \rho_m' e_{t-1}^m}{\sqrt{1-\|\rho_m\|^2}} - \frac{\alpha x}{\sqrt{1-\alpha^2}\sqrt{1-\|\rho_m\|^2}}\right)\varphi(x)dx}{1-\Phi(\tau\sqrt{1-\alpha^2})} & \text{if } s_{t-1} = 1 \end{cases}.$$

In Theorem 1, it is assumed that $\|\rho_m\| < 1$. In the case of $\|\rho_m\| = 1$, the transition probabilities are given by the following corollary.

**Corollary 1.** *If $\|\rho_m\| = 1$, the transition probability $\omega_\rho = \omega_\rho(s_{t-1}, s_{t-2})$ of $(s_t)$ to the state $s_t = 0$ conditional on the previous states and the past values of observed time series is given by*

*(a) If $\alpha = 0$, $\omega_\rho = 1\{\rho' e_t^m < \tau\}$.*

*(b) If $0 < \alpha < 1$,*

$$\omega_\rho = (1 - s_{t-1})\min\left(1, \frac{\Phi\left((\tau - \rho_m' e_{t-1}^m)\frac{\sqrt{1-\alpha^2}}{\alpha}\right)}{\Phi\left(\tau\sqrt{1-\alpha^2}\right)}\right)$$

$$+ s_{t-1}\max\left(0, \frac{\Phi\left((\tau - \rho_m' e_{t-1}^m)\frac{\sqrt{1-\alpha^2}}{\alpha}\right) - \Phi\left(\tau\sqrt{1-\alpha^2}\right)}{1 - \Phi\left(\tau\sqrt{1-\alpha^2}\right)}\right).$$

*(c) If* $-1 < \alpha < 0$,

$$\omega_\rho = (1 - s_{t-1})\min\left(1, \frac{1 - \Phi\left((\tau - \rho_m' e_{t-1}^m)\frac{\sqrt{1-\alpha^2}}{\alpha}\right)}{1 - \Phi\left(\tau\sqrt{1-\alpha^2}\right)}\right)$$
$$+ s_{t-1}\max\left(0, \frac{\Phi\left(\tau\sqrt{1-\alpha^2}\right) - \Phi\left((\tau - \rho_m' e_{t-1}^m)\frac{\sqrt{1-\alpha^2}}{\alpha}\right)}{\Phi\left(\tau\sqrt{1-\alpha^2}\right)}\right).$$

Given Theorem 1 and Corollary 1, the modified Markov switching filter from Chang, Choi, and Park (2017) can be used, with some obvious modifications, to estimate the parameters $A_m$,$B_m(0)$, $B_m(1)$, $\alpha$, $\rho_m$ and $\tau$ in the model and extract the latent factor $w_t$.

### 3.2 Identification of Structural Shocks

Suppose for now that we know the current regime $s_t$. We focus on the structural shocks which can be obtained from

$$e_t = B(s_t)^{-1}u_t$$

where $u_t$ is the reduced form error. Recall that $e_t$ is an $m$ dimensional vector. We use the correlation between the vector $e_t$ and the external shock $\epsilon_t$, which is a vector itself to obtain the response of the distribution of inflation expectations to the corresponding external shock.

Of course, the state variable $s_t$ is unobservable and the regime is unknown. However, the regime can be inferred and the inferred regime can be used for the regime dependent identification scheme used in the paper, the inferred regime can be used. From the filter used in the estimation, the inferred regime probability $\mathbb{P}(s_t|\mathcal{F}_t)$ of the regime is obtained for each regime. To minimize the error in inferring regimes, one may concentrate on sub-samples in which the probability of either regime is high. To this end, consider two real increasing sequences $a_T, b_T \to 1$ as $T \to \infty$, and define two sub-samples $S_0$ and $S_1$ of $\{1, 2, \cdots, T\}$ such that

$$S_0 = \left\{ t \middle| \mathbb{P}(s_t = 0 | \mathcal{F}_t) \geqslant a_T \right\}$$
$$S_1 = \left\{ t \middle| \mathbb{P}(s_t = 1 | \mathcal{F}_t) \geqslant b_T \right\}.$$

Since $a_T, b_T \to 1$ as $T \to \infty$, the probability of correctly inferring regimes increases and get close to one as the sample size increases. Therefore, using inferred regimes is allowed and should not affect the results in any significant manner, at least when the sample size is reasonably large.

# 4 Application: Expected Inflation Distribution (EID)

This paper uses data from the *Survey of Consumers* to estimate the density functions that will model the distribution of inflation expectations. In this section, we briefly describe the procedure we used to obtain the density functions from the survey data. This process will generate the functional time series of expected inflation densities. The analysis of the effects of economic policy on the distribution of inflation expectations is performed on the entire density function. Nevertheless, it is useful for the economic analysis to use different *aspects* of the distribution to describe its dynamics. Hence, from the density functions, we also extract the time series of some aspects of the distribution: mean, standard deviation, relative frequency at 2% (the Federal Reserve's inflation target) and at 0% (constant prices), the portion of the density that is negative and the tail mass (expectations that are more than three standard deviations away from the mean). These time series illustrate the different ways in which the entire distribution varies over time and aid the interpretation of the analysis.

## 4.1 From Survey Data to Functional Data

Each month the University of Michigan conducts the Survey of Consumers via telephone with a minimum of 500 participants[7]. Per design, the sample represents the population[8]. The survey consists of 50 core questions; the question of interest in this paper is: *By about what percent do you*

[7] With a range of 408 to 1205 valid responses for the inflation expectations question. See appendix.

[8] "The samples for the Surveys of Consumers are statistically designed to be representative of all American households, excluding those in Alaska and Hawaii." Taken from the Survey Description on their website.

*expect prices to go (up/down) on the average, during the next 12 months?* The estimation of the density functions follows from the responses to this question.

This paper uses a *non-parametric* method. This means that the shape of the density is not predetermined. Instead, the density is the weighted sum of kernel functions and the data is the main factor determining the shape of these densities. The name of this methodology is *kernel smoothing densities*. For further details see, for example, Bowman and Azzalini (1997).

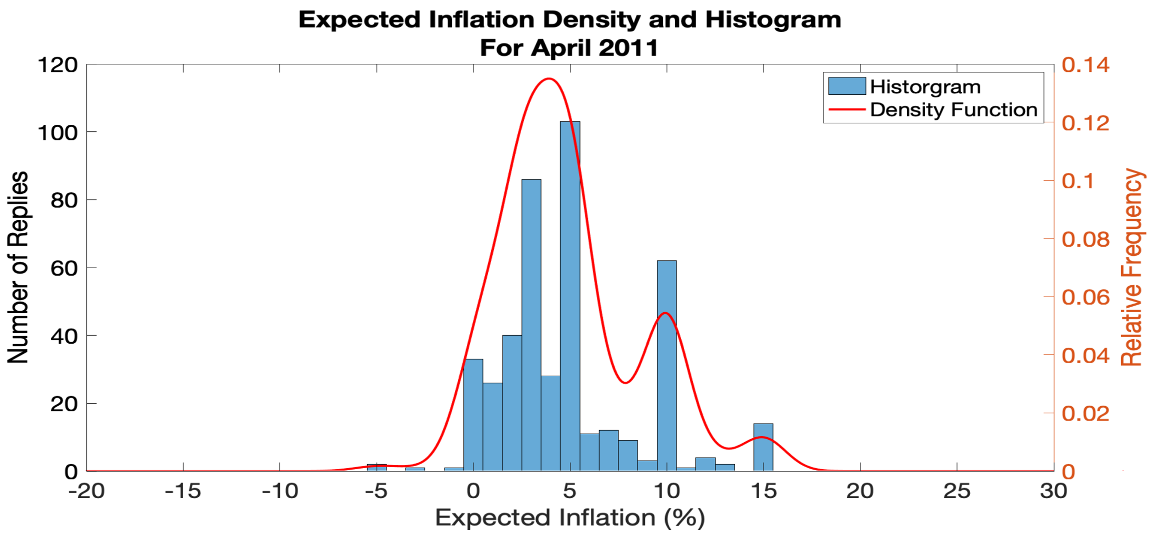


***Figure 6.*** *Time series of standard deviation corresponding to the expected inflation densities in the sample.*

To illustrate the process, Figure 6 displays the histogram of the single responses for April 2011. In the same figure, the red line is the resulting density from using a kernel smoothing estimation method. The estimation used a Epanechnikov kernel with the corresponding optimal bandwidth for each month[9] Using the density function provides an efficient way of representing all the data [10]. The precision of the estimation, that is, how close is the estimated density from the actual density (representing the whole population's inflation expectations), depends on the cross-sectional sample size of each month (Park and Qian, 2012).

Repeating this process for the whole sample generates the time series of inflation expectations densities. Figure 7 shows the times series for the given sample. The graph is "three-dimensional" given that the time series has the time axis from 1978 to 2021, the expected inflation axis from

[9]The bandwidth used for each month in the sample is available as a graph in the appendix.

[10]The implementation starts by describing the density functions using a wavelet-basis (Daubechies, 1992). The wavelets are an efficient way of *reading* the data on density functions.

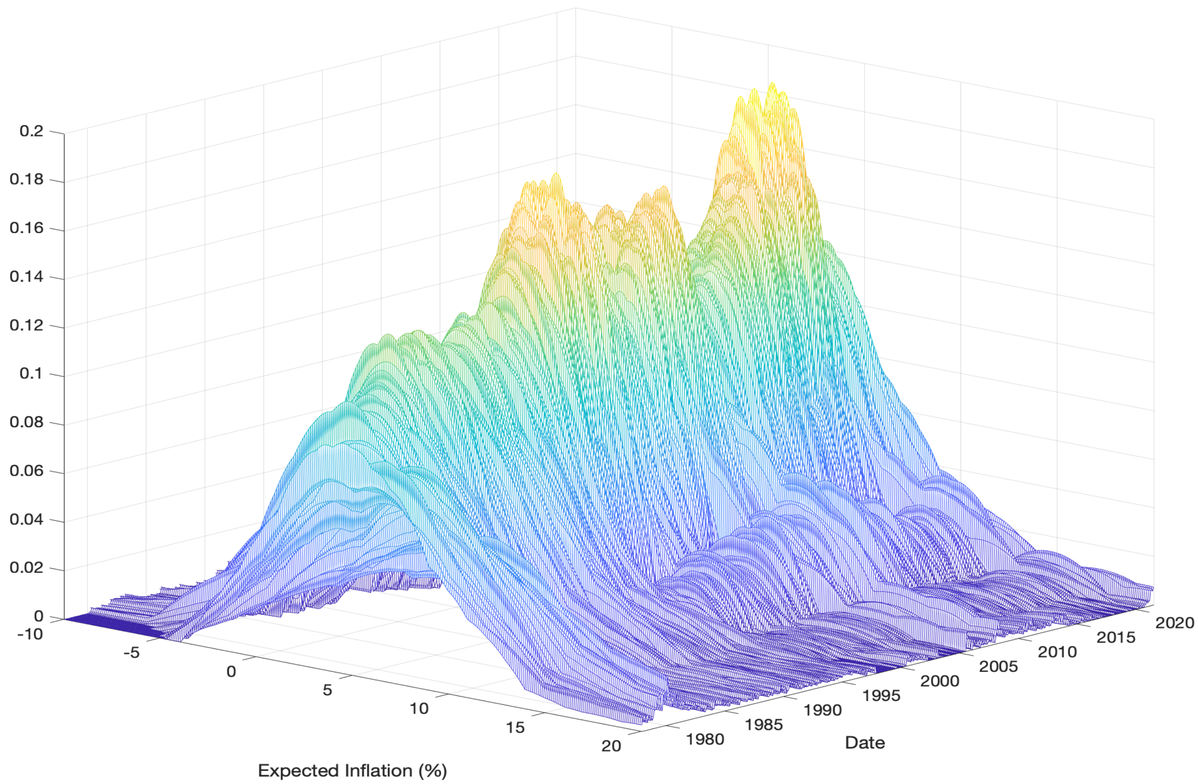


***Figure 7.*** *Functional time series of expected inflation densities. Sample from January 1978 to December 2021.*

-20% to 30%, and the value of the density. It represents the relative frequency of each inflation expectation.

For computational convenience we analyze the time series of *deviations* from the mean density. For illustration, Figure 8 displays the time series in a two dimensional plot. The function's interpretation is in terms of the relative size of the group of individuals with a particular inflation expectation. Negative values imply that fewer individuals have that expectation. On the contrary, positive values imply that more individuals, relative to the mean density, have the corresponding inflation expectation. One characteristic that is clear from the figure is that the most variation concentrate in the range of 0% to 15%.

It is helpful to describe the density function in terms of specific statistics. One may typically think of the mean or the standard deviation. The advantage of using densities is obtaining statistics that play a more significant role in economic analysis. For example, one can analyze the percentage

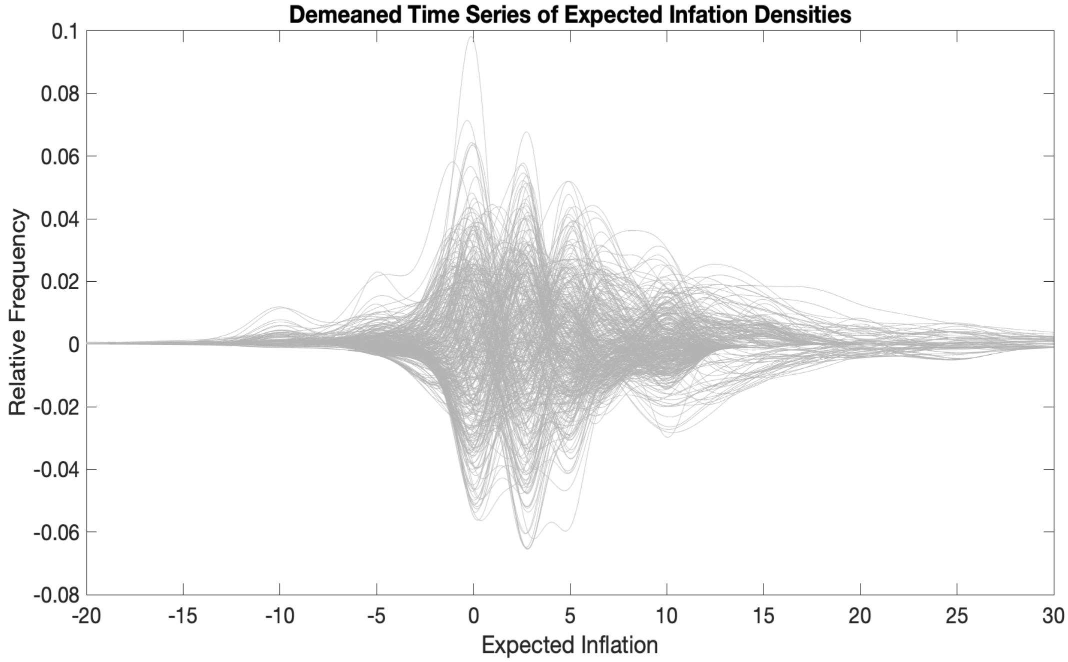


***Figure 8.*** *Functional time series of deviation of expected inflation densities from the mean density.*

of the population with inflation expectations "on-target" by calculating the density of expectation close to 2%.

## 4.2 Characterizing Expected Inflation Densities

The previous subsection explained the process that estimates density functions based on the survey data. This subsection provides additional tools to analyze and interpret density functions: distribution *statistics* or *aspects*.

Common *statistics* used to describe a density function are its moments. Mainly the mean and the variance (standard deviation). Additionally, one may think of other central tendency measures (median and mode), the range, or higher moments. One advantage of using density functions is the availability of other statistics. For example, as previously mentioned, a Central Bank with an inflation target of, say, 2% can be interested in analyzing the relative frequency of agents with inflation expectations "on target". Binder (2017) builds an index for economic uncertainty from people's tendency to reply round numbers (multiples of 5) for their inflation expectations. This phenomenon, is referred to *digit preference*, an additional *statistic* we may use for analysis.

Other literature studying the dispersion of inflation expectations uses the monthly interquartile or standard deviation. By using a density function means all properties or statistics described here and any other derivable from the density function, will be available throughout the analysis. Figure 9 shows example *properties* of expected inflation distributions.

Figure 9 shows the time series of six *statistics* for the time series of deviations of the distribution of inflation expectations from the mean distribution. In some time series more than in others, one observes that there are periods in which the observed *property* fluctuates more. This is the change in volatility that motivates the introduction of functional regime switching models in section 3. The

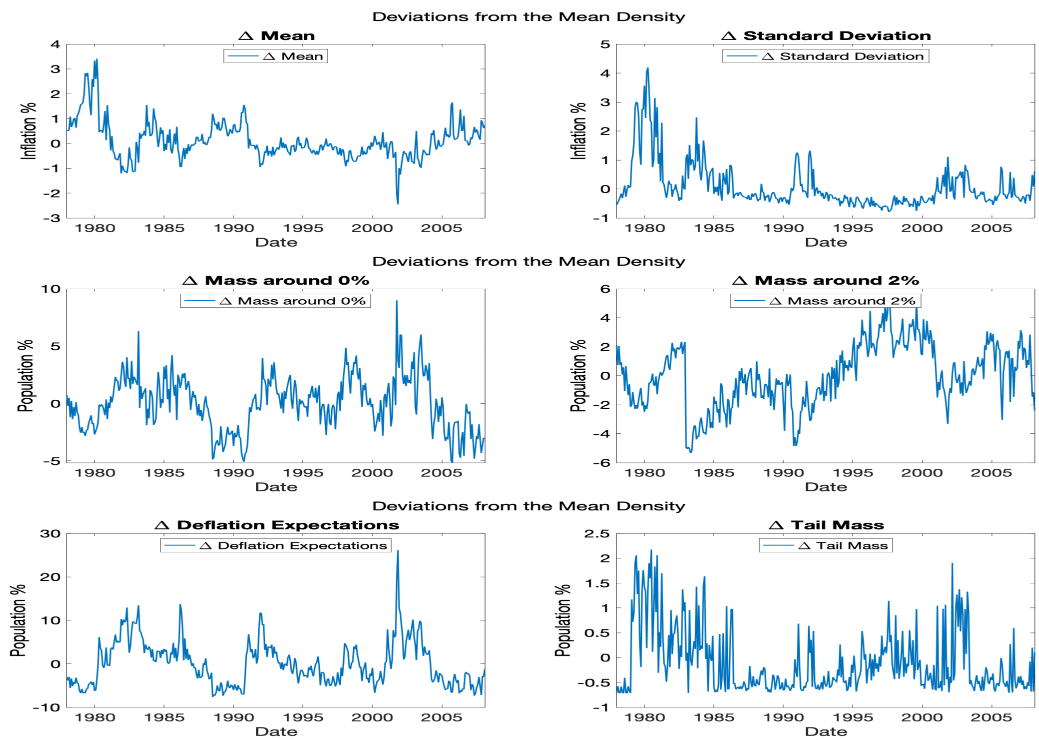


***Figure 9.*** *Time series of aspects describing the deviation of the distribution of inflation expectations from the mean. Sample January 1978 to December 2007.*

following information describes the definition of each of those *properties*:

(a) **Mean:** first moment of the density function.

(b) **Standard Deviation:** square-root of second moment of the density function.

(c) **Mass around 0%:** Portion of the population with inflation expectations within 1% of the most frequent expectation.

(d) **Mass around the 2%:** Portion of the population with inflation expectations between 1.5% and 2.5%.

(e) **Deflation expectations:** relative size of expectations below 0% (portion of individuals who expect prices to go down).

(f) **Tail Mass:** Percent of expectation over three-standard deviations away from the mean.

## 5 Regime Dependent Effects of Monetary and Fiscal Policy on EID

In this section, we present the results and analysis of the estimation of the model. We start by describing the regimes of the time series of expected inflation densities using the responses of the basis responses in each regime. Then, we analyze the regime dynamics of the process using the inferred probabilities and the parameters related to the latent factor. Finally, we analyze the effects of monetary and fiscal policy on the distribution of inflation expectations.

The estimation of the model used the sample from January 1978 to December 2007. The choice of this period is because the external shocks used to determine the effects of monetary and fiscal policy are only available until December 2007.

### 5.1 Description of the Regime using the Responses At-Impact

From the estimation we obtain two different covariance matrices $\Sigma(0)$ and $\Sigma(1)$. The way we obtain the impulse responses is by finding the linear combination of functional shocks that maximizes the correlation with the external shock. Since we are combining the shocks the factorization (identification) of the covariance matrices ultimately does not play a role. Therefore, we use recursive identification. So we use the factorization

$$\Sigma(s_t) = B(s_t)B(s_t)'$$

such that $B(s_t)$ is lower triangular. We use the columns of $B(s_t)$ to determine what we call basis responses. These are function which are used as basis to obtain the response of EID to external shocks using the correlation factors of each functional shock and the external shock. Figure 10 shows the at-impact basis responses of EID.

Given that we used three principal factors to describe the dynamics of the distribution of inflation expectations ($m = 3$) and according to the identification procedure described in the previous section we consider two external shocks for the structural analysis: the monetary policy shock estimated in Miranda-Agrippino and Ricco (2018) and a government defense spending shock from Auerbach and Gorodnichenko (2012).

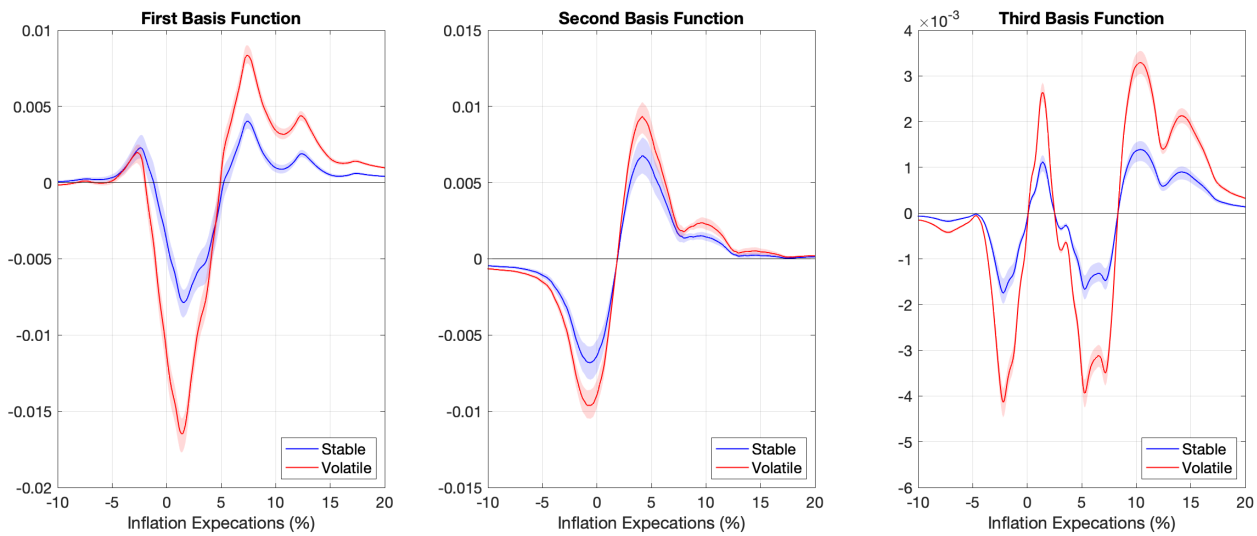


***Figure 10.*** *Regime Dependent Basis Responses.*

*Note:* These are the responses based on the columns of the lower triangular Cholesky factors of $\Sigma(s_t)$.

In figure 10 we can appreciate that during the volatile regime the same one-standard deviation shock causes a larger change in the distribution of inflation expectations. For implementation we focused the first eigenvalue of the covariance matrix of the reduced form errors to identify the model.

These responses are not the only way in which policy can generate different responses on the distribution of inflation expectations. The correlation of the external shocks with the functional shocks can also vary with the regime. So, in one regime a shock can have a high correlation with one of the shocks and low or even opposite sign correlation in another regime.

## 5.2 The Regime Dynamics

From the inferred probabilities in Figure 11 it is possible to observe that there is a lot of dynamics between the regimes. Periods around the recessions are typically the high volatility regime but other noticeable periods are of the volatile regime for example during 1987, 1994, 1996.

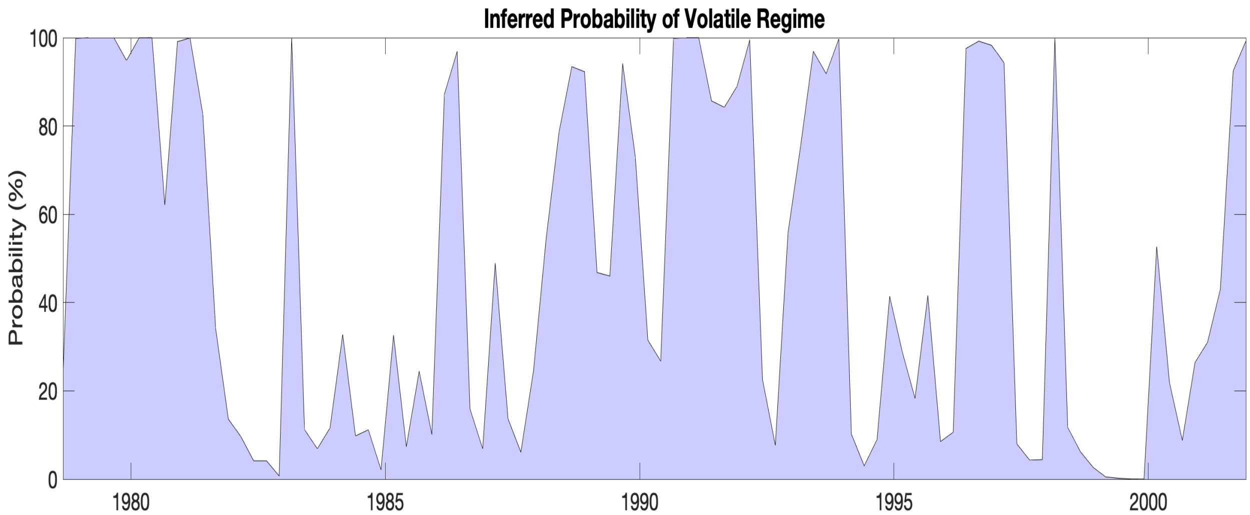


***Figure 11.*** *Inferred probability of a volatile regime.*

*Note:* For the estimation with the sample from 1978Q1 to 2007Q4 these are the inferred probabilities of a volatile regime.

| Parameter | $\alpha$ | $\tau$ | $\|\rho\|$ |
|---|---|---|---|
| Estimate | 0.8224 (0.0366) | $-0.625$ (0.3439) | 1 (0.1562) |

***Table 1.*** *Estimation of the regime dynamics parameters.*

*Note:* Standard errors obtained bootstrapping of the parameters that describe the regime dynamics.

From a the estimation of $\tau$ we learn that during the period of January 1978 to December 2019 both regime are observed equally often. the stable regime was observed more often. The value of estimated $\alpha$ indicates that the regimes are persistent.

The large value of the estimated norm of $\rho$, indicates strong endogenous feedback, which indicates that the transition probabilities are mainly determined by the same factors that influence the functional time series.

The parameter function $\rho$ (figure 12) tells us that when dispersion increases, so does the transition probability of to the volatile regime.

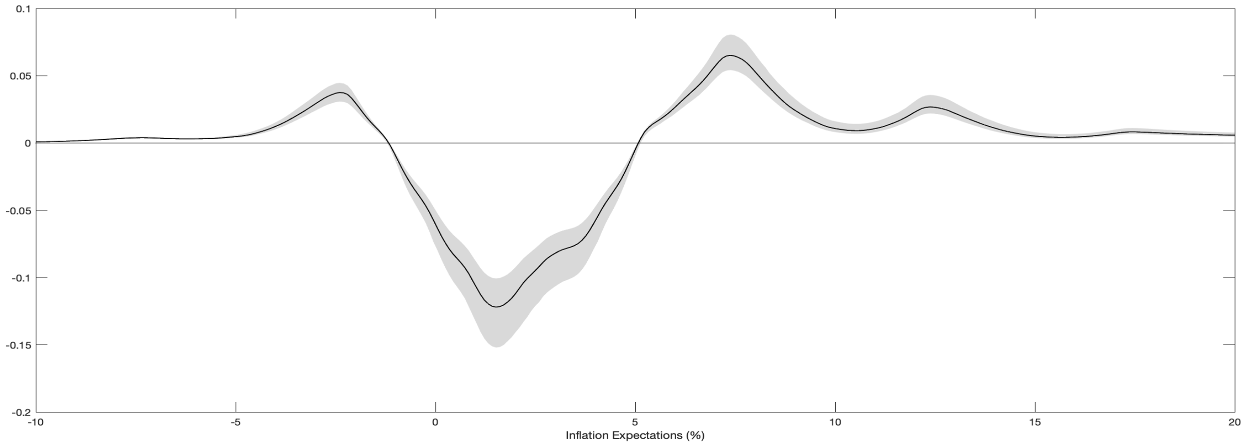


***Figure 12.*** *Parameter function $\rho$.*

*Note:* Parameter function $\rho$. The 68% confidence band is estimated using bootstrap methodology.

### 5.3 Regime Dependent Effects of Policy on EID

We want to determine the regime dependent effects of monetary and fiscal policy on EID. We first analyze the effects on the entire distribution and then the effects on selected statistics. We will demonstrate that modelling the distribution of inflation expectations as a regime-switching sheds light on some effect

**The effects of monetary policy.** Monetary policy has shifting effect. In the stable regime there is a significant reduction of inflation expectations in the range between 0% and 5% and also an increase in the interval around 10%. There is an increase of inflation expectations around 2% (inflation expectations anchoring). In the volatile regime the increase in inflation expectations is ever higher, not only negative inflation expectations but also low inflation expectations are less frequent after a contractionary monetary policy shock while we observe an increase in inflation expectations of high ($> 7\%$). The correlation between the monetary policy shock and EID functional shocks is $\underset{(10.85\%)}{21.96\%}$ in the stable regime and $\underset{(9.74\%)}{37.96\%}$ in the volatile regime. This percentage can be interpreted as the portion of the shock that is transmitted to EID.

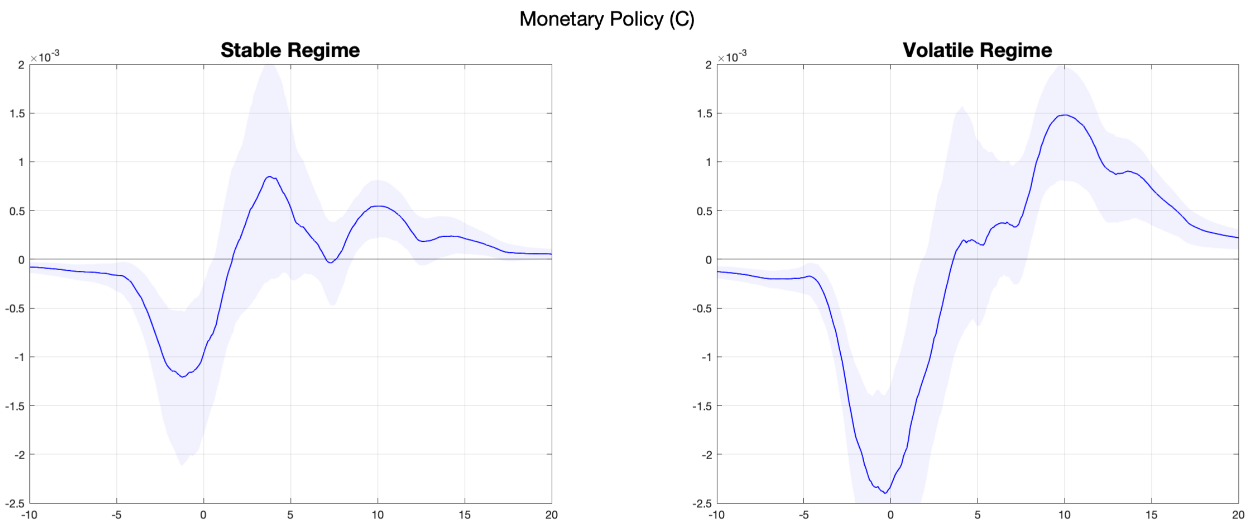


***Figure 13.*** *Response of the entire expected inflation density to a contractionary monetary policy shock.*

**The effects of fiscal policy.** The main economic effect of government spending on the distribution of inflation expectations is an increase on inflation expectations around 5% in the volatile regime. This increase, is not reflected in the mean or in the standard deviation of the distribution as one can see in figure 15 but is a significant difference that is not observable in the stable regime.

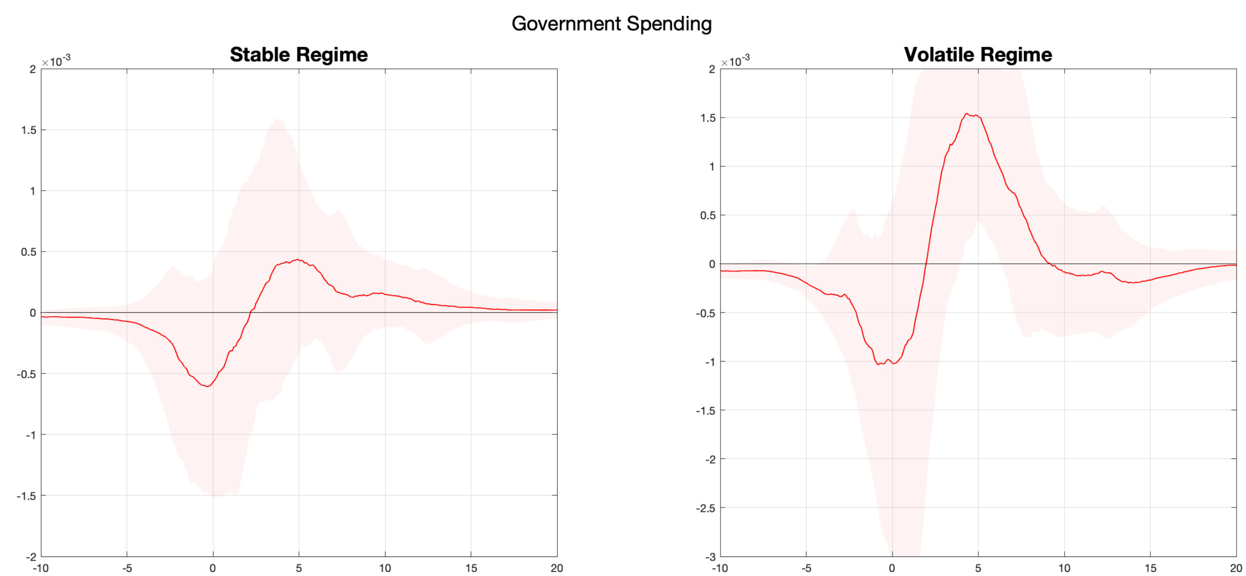


***Figure 14.*** *Response of the entire expected inflation density to a government defense spending shock.*

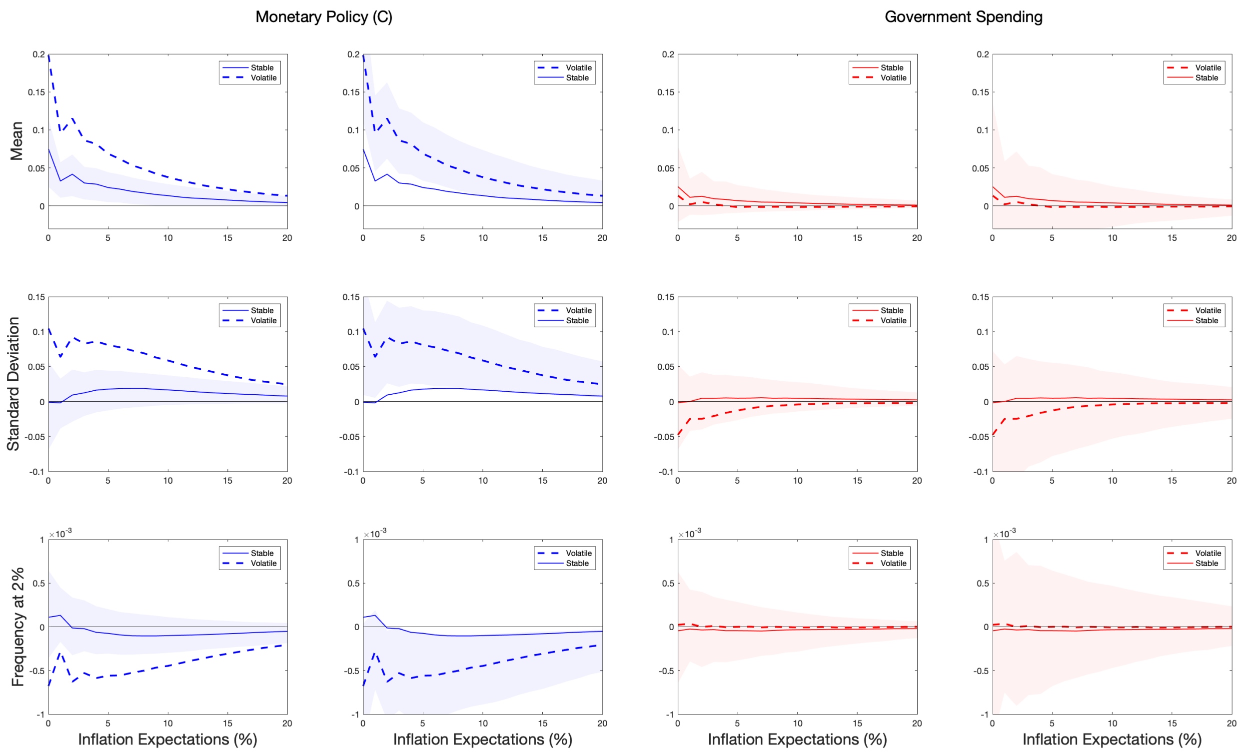


***Figure 15.*** *Regime dependent response of selected aspects of the distribution to monetary and fiscal policy shocks.*

# 6 Conclusion

We presented the implementation of a regime-switching model for functional time series. The critical step is to use functional principal components to represent the functional time series as vectors.

The model in this paper focuses on a regime-dependent covariance matrix or so-called volatility model; nonetheless, the approach we presented here allows for mean and autocovariance elements to switch regimes as needed.

As an example, this paper modeled the distribution of inflation expectations using density functions and determined how economic policy affects the distribution contingent on a regime. Here, we considered a monetary policy shock Miranda-Agrippino and Ricco (2018) and a defense government spending shock from Auerbach and Gorodnichenko (2012). We learned that Monetary policy increases inflation expectations about two times more during the volatile regime. This result implies that in a volatile regime, when the contractionary policy is trying to decrease inflation,

inflation expectations make this task more difficult; instead, during the stable regime, the effects on the mean and other aspects of the distribution are minor or none. This fact confirms that monetary policy has a better chance to be effective in periods of stability in the heterogeneity of inflation expectations.

Future research could study the effects of changes in the entire distribution of inflation expectations on inflation or real outcomes such as GDP or unemployment. Here the authors are considering a model with mixed elements (functional and scalar), which will improve our understating of the inflation expectations channel of policy.

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## A Analysis Based on a Linear Model

Based on a linear model (i.e. without changes in regime), here we present the impact of monetary and fiscal policy on the distribution of inflation expectations.

The model considered here is given by

$$\varphi_t = A\varphi_{t-1} + B\varepsilon_t \tag{10}$$

where $\varphi_t$ continues to be the demeaned expected inflation density. As opposed to the model in the main text the response-at-impact operator $B$ does not change in regime. The response at impact to a monetary policy and fiscal policy shock are shown in the following Figure.

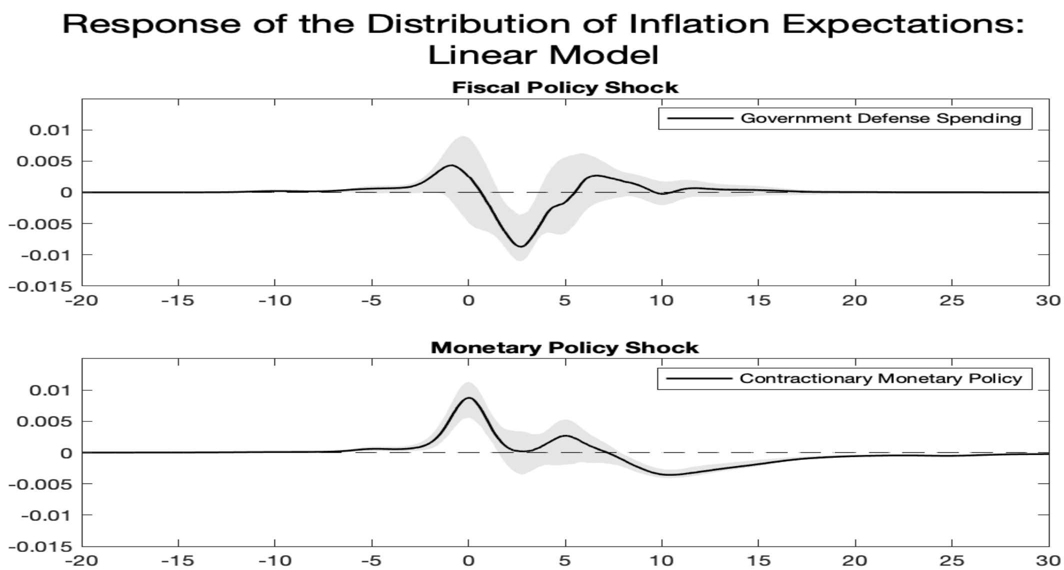


***Figure 16.*** *Response of the Distribution of Inflation Expectations to fiscal and monetary policy.*

Response to a monetary policy shock. Response to a government defense spending shock.

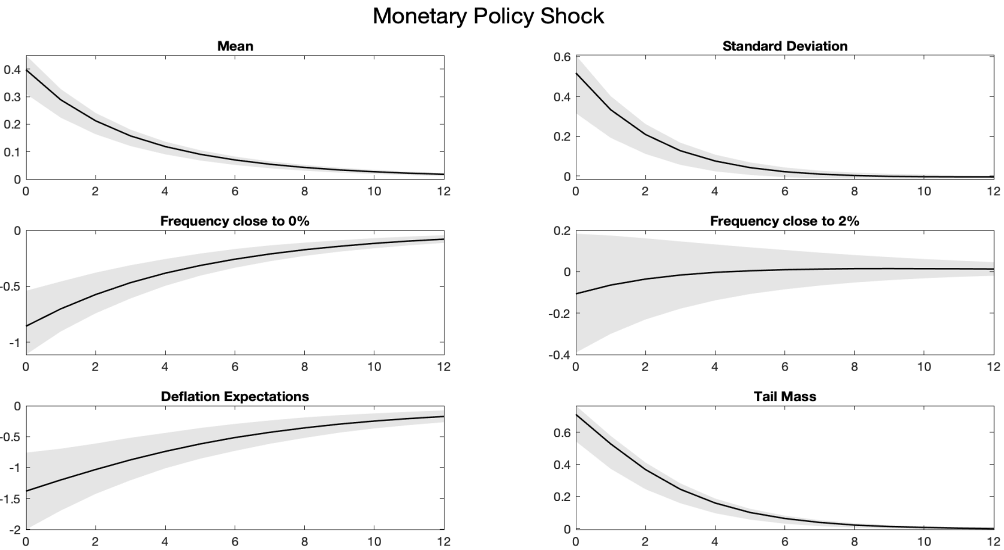


***Figure 17.*** *Impulse response functions of different aspects of how the distribution of inflation expectations deviate from the mean distribution after a fiscal policy shock.*

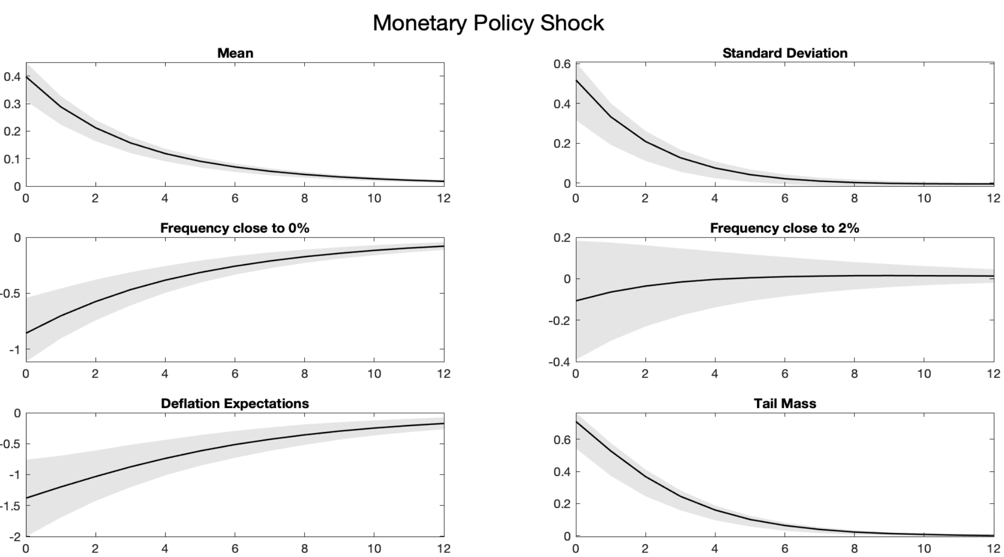


***Figure 18.*** *Impulse response functions of different aspects of how the distribution of inflation expectations deviate from the mean distribution after a monetary policy shock.*

# B Additional Information

## B.1 Bandwidth

In the main text, we mentioned that the bandwidth for estimating the density from the survey data depends on the specific data for the month. The following figure shows the bandwidth used in each month of the estimation.

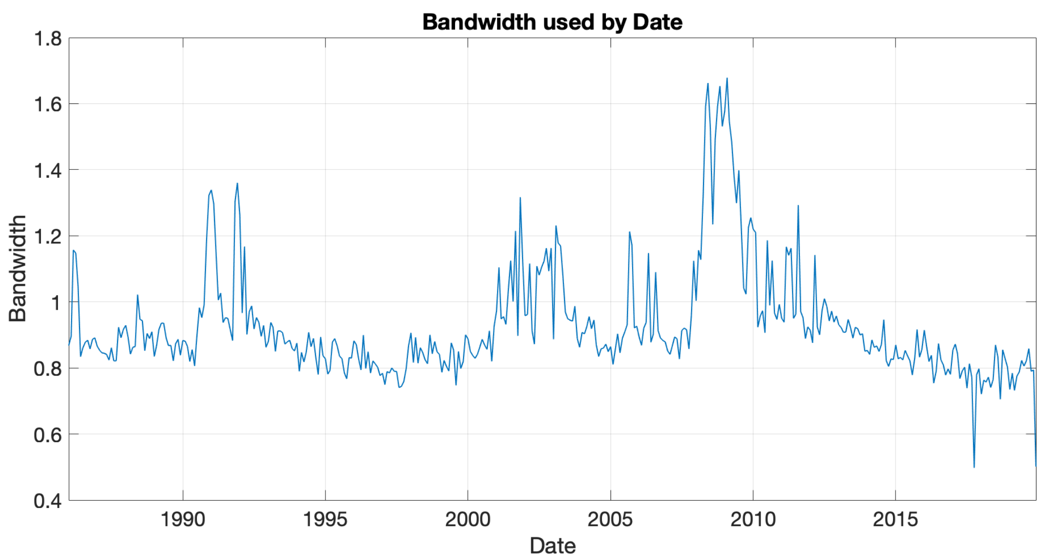


***Figure 19.*** *This is the bandwidth used in each month for the estimation using kernel smoothing densities.*

## B.2 Number of Responses

In the main text, we mentioned that the bandwidth for estimating the density from the survey data depends on the specific data for the month. The following figure shows the bandwidth used in each month of the estimation.

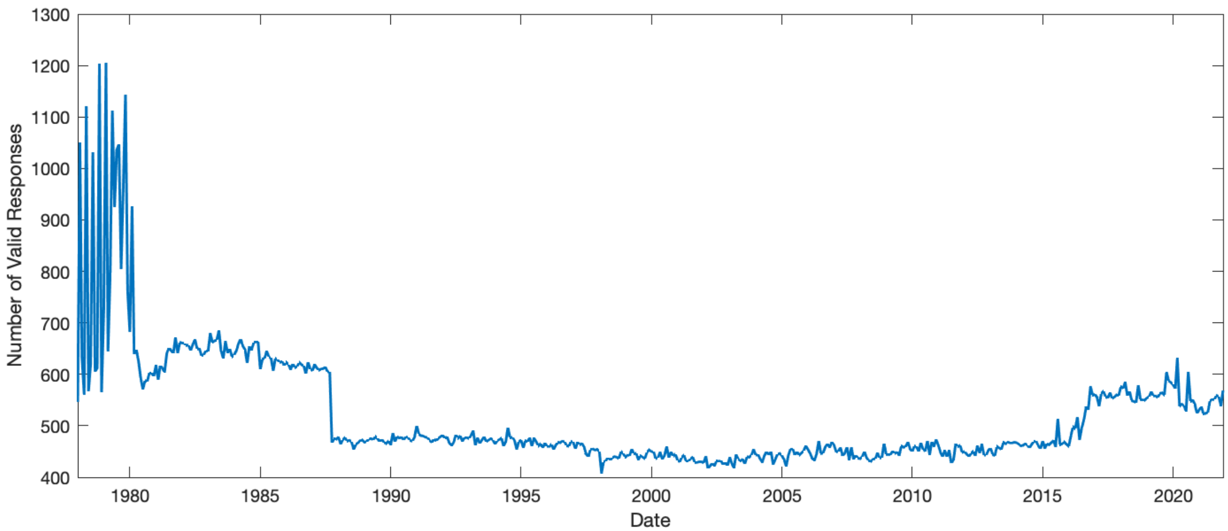


***Figure 20.*** *Number of valid responses in the University of Michigan's Survey of Consumers.*

# C Proofs

The following two proofs are adapted from Chang, Choi, and Park (2017) for the multivariate case in this paper.

**Proof of the Theorem:**

From equation (9′) follows that

$$\tilde{\eta}_t = \frac{w_t - \alpha w_{t-1} - \rho_m' \varepsilon_{t-1}^m}{\sqrt{1 - \|\rho_m\|^2}} \Rightarrow p(\tilde{\eta}_t | w_{t-1}, \varphi_{t-1}^m, \varphi_{t-2}^m) = \mathbb{N}(0, 1)$$

It follows that

$$\begin{aligned}
&\mathbb{P}\{w_t < \tau | w_{t-1}, w_{t-2}, \varepsilon_{t-1}^m, \varepsilon_{t-2}^m\} \\
&= \mathbb{P}\left\{ \tilde{\eta}_t < \frac{\tau - \alpha w_{t-1} - \rho_m' \varepsilon_{t-1}^m}{\sqrt{1 - \|\rho_m\|^2}} \middle| w_{t-1}, w_{t-2}, \varepsilon_{t-1}^m, \varepsilon_{t-2}^m \right\} \\
&= \Phi\left( \frac{\tau - \alpha w_{t-1} - \rho_m' \varepsilon_{t-1}^m}{\sqrt{1 - \|\rho_m\|^2}} \right)
\end{aligned}$$

Note that

$$p(w_t | w_{t-1}, w_{t-2}, \varepsilon_{t-1}^m, \varepsilon_{t-2}^m) = p(w_t | w_{t-1}, \varepsilon_{t-1}^m)$$

and that $w_{t-1}$ is independent of $\varepsilon_{t-1}^m$. Consequently, we have

$$\begin{aligned}
&\mathbb{P}(w_t < \tau | w_{t-1} < \tau, w_{t-2}, \varphi_{t-1}^m, \varphi_{t-2}^m) \\
&= \mathbb{P}(w_t < \tau | w_{t-1}\sqrt{1 - \alpha^2} < \tau\sqrt{1 - \alpha^2}, w_{t-2}, \varphi_{t-1}^m, \varphi_{t-2}^m) \\
&= \mathbb{P}(s_t = 0 | s_{t-1} = 0, w_{t-2}, \varphi_{t-1}^m, \varphi_{t-2}^m) \\
&= \frac{\int_{-\infty}^{\tau\sqrt{1-\alpha^2}} \Phi\left( \frac{\tau - \rho_m' \varepsilon_{t-1}^m}{\sqrt{1 - \|\rho_m\|^2}} - \frac{\alpha x}{\sqrt{1-\alpha^2}\sqrt{1 - \|\rho_m\|^2}} \right) \varphi(x) dx}{\Phi(\tau\sqrt{1 - \alpha^2})}
\end{aligned}$$

and

$$\mathbb{P}(w_t < \tau | w_{t-1} \geqslant \tau, w_{t-2}, \varepsilon_{t-1}^m, \varepsilon_{t-2}^m)$$

$$
\begin{aligned}
&= \mathbb{P}(w_t < \tau | w_{t-1}\sqrt{1-\alpha^2} \geqslant \tau\sqrt{1-\alpha^2}, w_{t-2}, \varepsilon^m_{t-1}, \varepsilon^m_{t-2}) \\
&= \mathbb{P}(s_t = 0 | s_{t-1} = 1, w_{t-2}, \varepsilon^m_{t-1}, \varepsilon^m_{t-2}) \\
&= \frac{\int_{\tau\sqrt{1-\alpha^2}}^{\infty} \Phi\left(\frac{\tau-\rho_m'\varepsilon^m_{t-1}}{\sqrt{1-\|\rho_m\|^2}} - \frac{\alpha x}{\sqrt{1-\alpha^2}\sqrt{1-\|\rho_m\|^2}}\right)\varphi(x)dx}{1-\Phi(\tau\sqrt{1-\alpha^2})}
\end{aligned}
$$

since in particular $w_{t-1}\sqrt{1-\alpha^2} =_d \mathbb{N}(0,1)$ from which the stated result from the transition density for $(s_t, \varepsilon^m_t)$ may be readily obtained.

Now, we write

$$
\begin{aligned}
p(s_t, \varepsilon^m_t | s_{t-1}, \cdots, s_1, \varepsilon^m_{t-1}, \cdots, \varepsilon^m_1) = p(\varepsilon^m_t | s_t, s_{t-1} \cdots, s_1, \varepsilon^m_{t-1}) \\
\times p(s_t | s_{t-1}, \cdots, s_1, \varepsilon^m_{t-1})
\end{aligned}
$$

It follows that

$$
p(\varepsilon^m_t | s_t, s_{t-1} \cdots, s_1, \varepsilon^m_{t-1}, \cdots, \varepsilon^m_1) = p(\varepsilon^m_t | s_t, s_{t-1}, \varepsilon^m_{t-1})
$$

Moreover, we have

$$
p(s_t | s_{t-1}, \cdots s_1, \varepsilon^m_{t-1}, \cdots \varepsilon^m_1) = p(s_t | s_{t-1}, \varepsilon^m_{t-1})
$$

as we have shown above. Therefore, it follows that

$$
\begin{aligned}
p(s_t, \varepsilon^m_t | s_{t-1}, \cdots, s_1, \varepsilon^m_{t-1}, \cdots, \varepsilon^m_1) &= p(\varepsilon^m_t | s_t, s_{t-1}, \varepsilon^m_{t-1}) \times p(s_t | s_{t-1}, \varepsilon^m_{t-1}) \\
&= p(s_t, \varepsilon^m_t | s_{t-1}, \varepsilon^m_{t-1})
\end{aligned}
$$

and $(s_t, \varepsilon^m_t)$ is a first order Markov process.□

**Proof of the Corollary:**

Consider here only the case $0 < \alpha < 1$. The proof for the case $\alpha = 0$ is trivial and the proof of $-1 < \alpha < 0$ can be easily done with a simple modification of the case of $0 < \alpha < 1$.
It follows that

$$\begin{aligned}
\mathbb{P}\{w_t < \tau | w_{t-1}, \varphi^m_{t-1}, \varphi^m_{t-1}\} &= \mathbb{P}\{\alpha w_{t-1} + \eta_t < \tau | w_{t-1}, \varphi^m_{t-1}, \varphi^m_{t-1}\} \\
&= \mathbb{P}\{\alpha w_{t-1} + \rho_m \varepsilon^m_{t-1} < \tau | w_{t-1}, \varphi^m_{t-1}, \varphi^m_{t-1}\} \\
&= \mathbb{P}\{\alpha w_{t-1} + \rho_m \varepsilon^m_{t-1} < \tau | w_{t-1}, \varepsilon^m_{t-1}\} \\
&= 1_{\{\alpha w_{t-1} + \rho'_m \varepsilon^m_{t-1} < \tau\}}
\end{aligned}$$

When $0 < \alpha < 1$

$$\begin{aligned}
\omega_\rho(s_{t-1} = 0, \varphi^m_{t-1}, \varphi^m_{t-2}) &= \mathbb{P}\left\{\alpha w_{t-1} + \rho'_m \varepsilon^m_{t-1} < \tau | w_{t-1} < \tau, \varepsilon^m_{t-1}\right\} \\
&= \mathbb{P}\left\{\sqrt{1-\alpha^2} w_{t-1} < \frac{\sqrt{1-\alpha^2}(\tau - \rho'_m \varepsilon^m_{t-1})}{\alpha} \middle| \sqrt{1-\alpha^2} w_{t-1} < \sqrt{1-\alpha^2}\tau, \varepsilon^m_{t-1}\right\} \\
&= \begin{cases} 1, & \text{if } \frac{1}{\alpha}(\tau - \rho'_m \varepsilon^m_{t-1}) < \tau \\ \frac{\Phi\left((\tau - \rho'_m \varepsilon^m_{t-1}) \frac{\sqrt{1-\alpha^2}}{\alpha}\right)}{\Phi\left(\tau\sqrt{1-\alpha^2}\right)}, & \text{otherwise} \end{cases}
\end{aligned}$$

similarly,

$$\begin{aligned}
\omega_\rho(s_{t-1} = 1, \varphi^m_{t-1}, \varphi^m_{t-2}) &= \mathbb{P}\left\{\alpha w_{t-1} + \rho'_m \varepsilon^m_{t-1} < \tau | w_{t-1} > \tau, \varepsilon^m_{t-1}\right\} \\
&= \mathbb{P}\left\{\sqrt{1-\alpha^2} w_{t-1} < \frac{\sqrt{1-\alpha^2}(\tau - \rho'_m \varepsilon^m_{t-1})}{\alpha} \middle| \sqrt{1-\alpha^2} w_{t-1} > \sqrt{1-\alpha^2}\tau, \varepsilon^m_{t-1}\right\} \\
&\frac{\Phi\left((\tau - \rho'_m \varepsilon^m_{t-1}) \frac{\sqrt{1-\alpha^2}}{\alpha}\right) - \Phi\left(\tau\sqrt{1-\alpha^2}\right)}{1 - \Phi\left(\tau\sqrt{1-\alpha^2}\right)} \times 1\left\{\frac{\tau - \rho'_m \varepsilon^m_{t-1}}{\alpha} \geqslant \tau\right\}
\end{aligned}$$

# D Bootstrapping

We will use bootstrapping to determine the confidence intervals of the statistics we estimate for the EID and its responses to external shocks. Start by consider a sample of the EID synchronized with the sample of the external shock. The majority of the external shocks we consider here are given in quarterly frequency so we consider the last observation of the EID for the corresponding quarter. That way, we have two identical samples in size and frequency.

In the following, we show how to generate copies of functional time series and the external shock of size $n$:

(a) Take the reduced form residuals from the estimated model: $\varepsilon_t$

(b) Use the corresponding matrix $B(s_t)$ to extract the structural shocks $e_t = B^{-1}(s_t)\varepsilon_t$. Use only periods for which the probability of the regime is high.

(c) Pick, with replacement, a sample of $T$ structural shocks $e_{b(1)}, e_{b(2)}, \cdots, e_{b(n)}$ from $\{e_1, e_2, \cdots\}$. Note that the latter set can be have less than $T$ elements if some periods are ambiguous in terms of the regime (not high enough probability for any regime).

(d) If an interaction with an external shock $\epsilon_t$ is of interest, the bootstrap copy of the external shock becomes $\epsilon_{b(1)}, \epsilon_{b(2)}, \cdots, \epsilon_{b(n)}$. This is to assure the preservation of contemporaneous effects of the external shock and the generated series.

(e) Define the new set of structural shocks $e_t^*$ by standardizing the series $e_{b(i)}$.

(f) Use $\alpha$ $\tau$, $\rho$, $e_{t-1}^*$ and a standard normal random number to generate an observation of the "latent" factor $w_t$.

(g) Generate a time series $(f_t)_t^*$ by $(f_t)^* = \widehat{(A)}(f_{t-1})^* + B(s_t)e_t^*$ using $y_0^{(m)*} = y_0^{(m)}$ and $s_t$ depending on the value of $w_t$.

(h) Use the basis $\{v_1^*, v_2^*, \cdots, v_m^*\}$ and the time series $(f_t)_t^*$ to generate a copy of the functional time series $f_t^*$.

(i) Use the demeaned functional time series $(f_t)^*$ and if applies the shocks from step 3 to estimate the statistics to be analyzed.